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Conservation of Energy, Entropy Identity, and Local Stability for the Spatially Homogeneous Boltzmann Equation

Xuguang Lu[1]

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For nonsoft potential collision kernels with angular cutoff, we prove that under the initial condition $f_0(v)(1+|v|^2+|\log f_0(v)|)\in L^1(\mathbf{R}^3)$, the classical formal entropy identity holds for all nonnegative solutions of the spatially homogeneous Boltzmann equation in the class $L^\infty([0,\infty); L^1_2(\mathbf{R}^3))\cap C^1([0,\infty); L^1(\mathbf{R}^3))$ [where $L^1_s(\mathbf{R}^3)=\{f \mid f(v)(1+|v|^2)^{s/2}\in L^1(\mathbf{R}^3)\}$], and in this class, the nonincrease of energy always implies the conservation of energy and therefore the solutions obtained all conserve energy. Moreover, for hard potentials and the hard-sphere model, a local stability result for conservative solutions (i.e., satisfying the conservation of mass, momentum, and energy) is obtained. As an application of the local stability, a sufficient and necessary condition on the initial data f_0 such that the conservative solutions f belong to $L^1_{\mathrm{loc}}([0,\infty); L^1_{2+\beta}(\mathbf{R}^3))$ is also given.

KEY WORDS: Boltzmann equation; conservation of energy; entropy identity; local stability.

1. INTRODUCTION

In this paper we study some fundamental properties of solutions of the spatially homogeneous Boltzmann equation

$$\frac{\partial}{\partial t} f(v,t)=Q(f,f)(v,t), \qquad (v,t)\in\mathbf{R}^3\times(0,\infty) \tag{1.1}$$

[1] Department of Applied Mathematics, Tsinghua University, Beijing 100084, People's Republic of China; e-mail: xglu@math.tsinghua.edu.cn.

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which describes the time evolution of the velocity distribution f of a spatially homogeneous dilute gas of particles. In equation (1.1), Q is the collision operator acting on functions of velocity v:

$$Q(f, f)(v) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)[f(v')\, f(v'_*) - f(v)\, f(v_*)]\, d\omega\, dv_*$$

(in (1.1) $Q(f, f)(v, t)$ means $Q(f(\cdot, t), f(\cdot, t))(v)$) which describes the rate of change of f due to a binary collision. Here v, v_* are the velocities of two particles before they collide, and v', v'_* are their velocities after the collision. Let us first recall some basic facts about equation (1.1) which are also used later. According to the conservation laws of momentum and kinetic energy, v', v'_* and v, v_* have the relations: $v' + v'_* = v + v_*$, $|v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2$ which are equivalent to the explicit representation:

$$v' = v - \langle v - v_*, \omega \rangle\, \omega, \qquad v'_* = v_* + \langle v - v_*, \omega \rangle\, \omega, \qquad \omega \in \mathbf{S}^2 \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbf{R}^3$, $|v|^2 = \langle v, v \rangle$ and $\mathbf{S}^2 = \{\omega \in \mathbf{R}^3 \mid |\omega| = 1\}$. The collision kernel $B(z, \omega)$ is a nonnegative Borel function of $|z|$ and $|\langle z, \omega \rangle|$ only. For the interaction potentials of inverse power laws, $B(z, \omega)$ takes the form (see, e.g., refs. 6 and 18):

$$B(z, \omega) = b(\theta)\, |z|^\beta, \qquad \theta = \arccos(|z|^{-1}\, |\langle z, \omega \rangle|), \qquad -1 < \beta \leqslant 1 \tag{1.3}$$

The exponent β is related to the potentials of interacting particles, namely, the soft potentials ($-1 < \beta < 0$), the Maxwell model ($\beta = 0$), the hard potentials ($0 < \beta < 1$) and the hard sphere model ($\beta = 1$, $b(\theta) = \text{const.} \times \cos(\theta)$). Kernels with nonsoft potentials satisfy the following form:

$$B(z, \omega) \leqslant b(\theta)(1 + |z|^\beta), \qquad \theta = \arccos(|z|^{-1} |\langle z, \omega \rangle|), \qquad 0 \leqslant \beta \leqslant 1 \tag{1.4}$$

The nonnegative function $b(\theta)$ is often assumed to satisfy the angular cutoff condition:

$$0 < A_0 := 4\pi \int_0^{\pi/2} b(\theta) \sin(\theta)\, d\theta < \infty \tag{1.5}$$

so that the collision operator can be written as the difference of the gain operator Q^+ and the loss operator Q^-: $Q(f, f) = Q^+(f, f) - Q^-(f, f)$,

$$Q^+(f, f)(v) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\, f(v')\, f(v'_*)\, d\omega\, dv_*,$$

$$Q^-(f, f)(v) = f(v) \int_{\mathbf{R}^3} \left[\int_{\mathbf{S}^2} B(v - v_*, \omega)\, d\omega \right] f(v_*)\, dv_*$$

Solutions (as the velocity distribution functions) of the initial-value problem for (1.1) should be nonnegative on $\mathbf{R}^3 \times [0, \infty)$. Their initial data $f|_{t=0} = f_0$ are usually assumed to satisfy the conditions:

$$f_0 \geqslant 0, \qquad 0 < \int_{\mathbf{R}^3} f_0(v)(1 + |v|^2 + |\log f_0(v)|)\, dv < \infty \tag{1.6}$$

i.e., such that the mass and energy are finite and the entropy can be defined initially. The rigorous treatment for the solutions of (1.1) are usually established in function classes (for instance) $L^\infty([0, \infty); L^1_s(\mathbf{R}^3)) \cap C^1([0, \infty); L^1(\mathbf{R}^3))$ where $s \geqslant 0$ and $L^1_s(\mathbf{R}^3)$ are defined by

$$f \in L^1_s(\mathbf{R}^3) \Leftrightarrow \|f\|_{L^1_s} := \int_{\mathbf{R}^3} |f(v)|(1 + |v|^2)^{s/2}\, dv < \infty$$

In such classes, the initial-value problem for equation (1.1) is written (see, e.g., ref. 12)

$$\frac{d}{dt} f(\cdot, t) = Q(f, f)(\cdot, t) \qquad \text{in} \quad L^1(\mathbf{R}^3), \quad t \in [0, \infty); \quad f|_{t=0} = f_0 \tag{1.7}$$

The derivative in (1.7) is of course defined with the norm $\|\cdot\|_{L^1}$. Here as usual we do not distinguish between $f(v, t)$ and its modifications on v-null sets. That is, in the class $C([0, \infty); L^1(\mathbf{R}^3))$, $f = g$ means $\|f(\cdot, t) - g(\cdot, t)\|_{L^1} \equiv 0$ on $[0, \infty)$. In this sense, every function $f \in C([0, \infty); L^1(\mathbf{R}^3))$ is a measurable function on $\mathbf{R}^3 \times [0, \infty)$. For the nonnegative solutions of (1.7), the most interesting class is $L^\infty([0, \infty); L^1_2(\mathbf{R}^3)) \cap C^1([0, \infty); L^1(\mathbf{R}^3))$ (i.e., $s = 2$) because in this class the mass and energy are bounded in time. It can be shown that if the collision kernel $B(z, \omega)$ satisfies (1.4) and (1.5), then the nonnegative solution $f \in L^\infty([0, \infty); L^1_2(\mathbf{R}^3)) \cap C^1([0, \infty); L^1(\mathbf{R}^3))$ of equation (1.7) is equivalent to the solution of the integral equation

$$f(v, t) = f_0(v) + \int_0^t Q(f, f)(v, \tau)\, d\tau, \qquad t \in [0, \infty), \quad v \in \mathbf{R}^3 \backslash Z \tag{1.8}$$

for nonnegative measurable functions f on $\mathbf{R}^3\times[0,\infty)$ which satisfy

$$f\in L^\infty([0,\infty);L^1_2(\mathbf{R}^3)) \quad\text{and}\quad Q^\pm(f,f)(v,t)\in L^1_{\rm loc}([0,\infty)),\quad \forall v\in\mathbf{R}^3\backslash Z \tag{1.9}$$

where Z is a v-null set independent of t. (See refs. 2, 18, 7 and 12.) Therefore, throughout this paper, whenever saying that f is a solution of the Boltzmann equation (1.1), it always means that f is a nonnegative solution of the equation (1.7), or equivalently, of the equation (1.8) with (1.9). Moreover, for convenience of the present paper, a solution of the equation (1.1) in the class $L^\infty([0,\infty);L^1_2(\mathbf{R}^3))\cap C^1([0,\infty);L^1(\mathbf{R}^3))$ will be called a *conservative solution* if it conserves the mass, momentum and energy on the whole closed interval $[0,\infty)$ of time.

So far the mathematical results on the spatially homogeneous Boltzmann equation are rather complete (in comparison with the inhomogeneous Boltzmann equation). Here we summarize some results related to the present paper. Especially we only mention the cases of nonsoft potential collision kernels (1.3) $(0\leqslant\beta\leqslant1)$ and (1.4) with angular cutoff (1.5).

(i) (Arkeryd[1]). Assume that the collision kernel $B(z,\omega)$ satisfy (1.4) and (1.5). If f_0 satisfies (1.6), then there exists a nonnegative solution f of the Boltzmann equation (1.1) in the class $L^\infty([0,\infty);L^1_2(\mathbf{R}^3))\cap C^1([0,\infty);L^1(\mathbf{R}^3))$ such that $f|_{t=0}=f_0$ and f satisfies the conservation of mass and momentum:

$$\int_{\mathbf{R}^3} f(v,t)(1,v)\,dv=\int_{\mathbf{R}^3} f_0(v)(1,v)\,dv,\qquad t\in[0,\infty)$$

and the nonincrease of energy:

$$\int_{\mathbf{R}^3} f(v,t)\,|v|^2\,dv\leqslant\int_{\mathbf{R}^3} f_0(v)\,|v|^2\,dv,\qquad t\in[0,\infty) \tag{1.10}$$

Moreover, the solution f can be also chosen such that it holds the entropy inequality (DiPerna and Lions[9]):

$$H(f)(t)\leqslant H(f_0)-\frac{1}{4}\int_0^t d\tau\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)$$
$$\times(f'f'_*-ff_*)\log\left(\frac{f'f'_*}{ff_*}\right)d\omega\,dv_*\,dv$$

where

$$H(f)(t) = \int_{\mathbf{R}^3} f(v, t) \log f(v, t)\, dv, \qquad t \in [0, \infty),$$

$$f = f(v, \cdot), \quad f_* = f(v_*, \cdot), \quad f' = f(v', \cdot), \quad f'_* = f(v'_*, \cdot)$$

It is obvious that under the cutoff conditions (1.4) and (1.5), every nonnegative solution f of Eq. (1.1) in the class $L^\infty([0, \infty); L^1_2(\mathbf{R}^3)) \cap C^1([0, \infty); L^1(\mathbf{R}^3))$ always conserves mass and momentum, and for $\beta = 0$, f is also a unique conservative solution.

(ii) Let the kernel $B(z, \omega)$ be given by (1.3) and (1.5) with $0 < \beta \leqslant 1$. If in addition to (1.6), the initial datum $f_0 \in L^1_{s_1}(\mathbf{R}^3)$ for some $s_1 > 2$, then the solution f in part (i) can be chosen with the equality in (1.10) (i.e., f is a conservative solution) and such that $f \in L^\infty([0, \infty); L^1_{s_1}(\mathbf{R}^3)) \cap C^1([0, \infty); L^1(\mathbf{R}^3)$ (Elmroth[10]), and has the property of moment production (Desvillettes[8]): for any $t_0 > 0$ and any $s > s_1$, $f \in L^\infty([t_0, \infty); L^1_s(\mathbf{R}^3)) \cap C^1([t_0, \infty); L^1_s(\mathbf{R}^3))$. Moreover, in the same class $L^\infty([0, \infty); L^1_{s_1}(\mathbf{R}^3)) \cap C^1([0, \infty); L^1(\mathbf{R}^3))$ $(s_1 > 2)$, the conservative solution f is also unique (Wennberg[20] and Gustafsson[11]), and, if the function $b(\theta)$ in (1.5) is also locally bounded on the open interval $(0, \pi/2)$, then for any $s \geqslant 0$, the solution f converges strongly in $L^1_s(\mathbf{R}^3)$ towards the equilibrium as $t \to \infty$ (Gustafsson[12] and Wennberg[19]).

(iii) (Wennberg[21]). Let the kernel $B(z, \omega)$ be the same as in part (ii). Then the solution f in part (i) can be also chosen such that for any $s > 2$ there exist positive constants a, b such that

$$\int_{\mathbf{R}^3} f(v, t)\, |v|^s\, dv \leqslant \left[\frac{b}{1 - \exp(-at)} \right]^{s/\beta}, \qquad t > 0 \tag{1.11}$$

Moment estimates like (1.11) are important; some applications of such estimates have been given by (for instance) Bobylev[3] (see also below). It should be noted that the estimate (1.11) together with the nonincrease of energy (1.10) imply the conservation of energy, so that f is actually a conservative solution. In fact by taking $s \geqslant 3$, (1.11) implies that the energy is conserved on every subinterval $[1/n, \infty)$: $\int_{\mathbf{R}^3} f(v, t)\, |v|^2\, dv = \int_{\mathbf{R}^3} f(v, 1/n)\, |v|^2\, dv$, $t \geqslant 1/n$, $n = 1, 2, \ldots$. Since for almost all $v \in \mathbf{R}^3$, $t \mapsto f(v, t)$ is continuous on $[0, \infty)$, it follows from Fatou's Lemma and the nonincrease of energy (1.10) that

$$\int_{\mathbf{R}^3} f(v, 0)\, |v|^2\, dv \leqslant \lim_{n \to \infty} \int_{\mathbf{R}^3} f(v, 1/n)\, |v|^2\, dv \leqslant \int_{\mathbf{R}^3} f(v, 0)\, |v|^2\, dv$$

Therefore the energy is conserved on whole $[0, \infty)$.

(iv) (Uniqueness of conservative solutions). Recently, Mischler, and Wennberg (ref. 15) proved that for the hard potentials (1.3) ($0<\beta\leqslant 1$) and (1.5), if the initial datum only satisfies $0\leqslant f_0\in L^1_2(\mathbf{R}^3)$, i.e., the mass and energy are finite initially, the conservative solution of the Boltzmann equation (1.1) still exists and is unique. Their proof of the uniqueness is based on a moment production property (obtained in their same paper):

$$\|f(\cdot,t)\|_{L^1_{2+\beta}}=\int_{\mathbf{R}^3} f(v,t)(1+|v|^2)^{(2+\beta)/2}\,dv$$
$$\leqslant\frac{\delta(t)}{t}\qquad\text{with}\quad \delta(t)\to 0(t\to 0^+)\tag{1.12}$$

and the following Nagumo's uniqueness criterion:

Let $u(t)$ be a nonnegative measurable function on $[0,T]$ satisfying for some positive constant $\lambda<1$,

$$u(0)=0,\qquad \sup_{0<t\leqslant T}\frac{u(t)}{t}<\infty,\qquad u(t)\leqslant\lambda\int_0^t\frac{u(\tau)}{\tau}d\tau\,,\qquad t\in(0,T]$$

Then $u(t)\equiv 0$ on $[0,T]$.

This uniqueness criterion is easily shown: We have $\sup_{0<t\leqslant T}(u(t)/t)$ $\leqslant\lambda\sup_{0<t\leqslant T}(u(t)/t)$ which implies $u(t)\equiv 0$ on $[0,T]$. In the proof of the uniqueness (ref. 15), the function $u(t)$ is taken $\|f(\cdot,t)-g(\cdot,t)\|_{L^1_2}$ where f and g are both conservative solutions of (1.1) with the same initial datum: $f|_{t=0}=g|_{t=0}$.

Our results of the present paper show that for the nonsoft potential (1.4) and (1.5) and under the only initial condition (1.6), the classical formal entropy identity (i.e., the formula (2.1) below) does actually hold for all solutions of the Boltzmann equation (1.1) in the class $L^\infty([0,\infty);L^1_2(\mathbf{R}^3))\cap C^1([0,\infty);L^1(\mathbf{R}^3))$; and in this class, the nonincrease of energy (1.10) always implies the conservation of energy for all solutions (Theorem 1). Furthermore, it is shown that for hard potentials (1.3) and (1.5) with $0<\beta\leqslant 1$, a local stability property holds for all conservative solutions (Theorem 2). As an application of the local stability, we give a sufficient and necessary condition on the initial data f_0 such that the conservative solutions f belong to $L^1_{\mathrm{loc}}([0,\infty);L^1_{2+\beta}(\mathbf{R}^3))$ (Theorem 3). Detail statements of Theorems 1–3 and the proof about the conservation of energy are given in Section 2. The proof of the entropy identity is given in Section 3. In Section 4, we give an important improvement of the Wennberg's estimate (1.11) (Theorem 4). Then, as an application of the new estimate, we prove in Section 5 the local stability of conservative solutions. The proof of Theorem 3 is given in Section 6.

2. THEOREMS 1–3 AND THE PROOF OF AN ENERGY EQUALITY

Theorem 1. Assume that the collision kernel $B(z, \omega)$ satisfy (1.4) and (1.5). Let f_0 be an initial datum satisfying (1.6), and let f be any non-negative solution of the Boltzmann equation (1.1) in the class $L^{\infty}([0, \infty); L^1_2) \cap C^1([0, \infty); L^1(\mathbf{R}^3))$ with $f|_{t=0} = f_0$. Then $\sup_{t \geqslant 0} \int_{\mathbf{R}^3} f(v, t) \times |\log f(v, t)| \, dv < \infty$ and for all $t \in [0, \infty)$ we have the entropy identity:

$$H(f)(t) = H(f_0) - \frac{1}{4} \int_0^t d\tau \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega) \times (f'f'_* - ff_*) \log\left(\frac{f'f'_*}{ff_*}\right) d\omega \, dv_* \, dv \tag{2.1}$$

and the nondecrease of energy:

$$\int_{\mathbf{R}^3} f(v, t) \, |v|^2 \, dv \geqslant \int_{\mathbf{R}^3} f_0(v) \, |v|^2 \, dv, \qquad t \in [0, \infty) \tag{2.2}$$

More precisely, we have for all $t \in [0, \infty)$

$$\int_{\mathbf{R}^3} f(v, t) \, |v|^2 \, dv = \int_{\mathbf{R}^3} f_0(v) \, |v|^2 \, dv + \lim_{\varepsilon \to 0+} \int_0^t d\tau \iint_{\mathbf{R}^3 \times \mathbf{R}^3} K_{\varepsilon}(v, v_*) \, f(v, \tau) \, f(v_*, \tau) \, dv_* \, dv \tag{2.3}$$

where

$$K_{\varepsilon}(v, v_*) = \frac{1}{2\varepsilon} \int_{\mathbf{S}^2} B(v - v_*, \omega) \log\left(1 + \frac{\varepsilon^2 \, |v'|^2 \, |v'_*|^2}{1 + \varepsilon(|v|^2 + |v_*|^2)}\right) d\omega, \qquad \varepsilon > 0$$

As a consequence, the entropy $-H(f)(t)$ is absolutely continuous on $[0, T]$ $(\forall T > 0)$ and for almost all $t \in (0, \infty)$

$$-\frac{d}{dt} H(f)(t) = \frac{1}{4} \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega) \times (f'f'_* - ff_*) \log\left(\frac{f'f'_*}{ff_*}\right) d\omega \, dv_* \, dv$$

Furthermore, if the solution f satisfies the nonincrease of energy (1.10), then the energy is conserved:

$$\int_{\mathbf{R}^3} f(v,t)\,|v|^2\,dv = \int_{\mathbf{R}^3} f_0(v)\,|v|^2\,dv, \qquad t \in [0,\infty)$$

Theorem 2. Assume the collision kernel $B(z,\omega)$ satisfy (1.3) and (1.5) with $0<\beta\leqslant 1$. Let f_0 satisfy (1.6), and let f be a conservative solution of the Boltzmann equation (1.1) with $f|_{t=0}=f_0$. Then there is a continuous increasing function $\Phi_f(\cdot)$ on $[0,\infty)$ satisfying $\Phi_f(0)=0$ and depending only on f, $b(\cdot)$ and β, such that for all conservative solutions g of (1.1),

$$\|g(\cdot,t)-f(\cdot,t)\|_{L^1_2} \leqslant \Phi_f(\|g_0-f_0\|_{L^1_2})\,e^{ct}, \qquad t\in[0,\infty) \tag{2.4}$$

where $g_0=g|_{t=0}$, c is a positive constant depending only on f_0, $b(\cdot)$ and β.

Remarks. 1. Under the conditions in Theorem 2, the continuous increasing function $\Phi_f(\cdot)$ can be taken as (see the proof of Theorem 2)

$$\Phi_f(r)=C[r+\sqrt{r}+\Psi_f(r)], \qquad r\in[0,\infty)$$

where C is a positive constant depending only on f_0, $b(\cdot)$ and β, and $\Psi_f(\cdot)$ is defined by

$$\Psi_f(r)=\sup_{0\leqslant t\leqslant r}\int_{|v|>1/\sqrt{r}} f(v,t)(1+|v|^2)\,dv, \qquad r>0;\quad \Psi_f(0)=0 \tag{2.5}$$

2. From our proof of Theorem 2 (in Section 3) one may find that the moment production property as in (1.12), or even as

$$\|f(\cdot,t)\|_{L^1_{2+\beta}} \leqslant C[1+(1/t)^{1+\varepsilon}], \qquad t>0,\quad 0<\varepsilon<1,$$

is enough to obtain the local stability result for conservative solutions; the Nagumo's uniqueness criterion does not work on such stability.

Theorem 3. Assume that the collision kernel $B(z,\omega)$ satisfy (1.3) and (1.5) with $0<\beta\leqslant 1$. Let f_0 satisfy (1.6) and let f be a conservative solution of the equation (1.1) with $f|_{t=0}=f_0$. Then the following (2.6) and (2.7) are equivalent:

$$\int_{\mathbf{R}^3} f_0(v)\,|v|^2 \log^+ |v|\,dv < \infty \tag{2.6}$$

$$\int_0^T dt \int_{\mathbf{R}^3} f(v,t)(1+|v|^2)^{(2+\beta)/2}\,dv < \infty, \qquad \forall T > 0 \tag{2.7}$$

Our proof for Theorem 1 is divided into two parts. The first part given below is the proof of the energy equality (2.3), the second part is the proof of the entropy identity (2.1) given in the next section.

Proof of the Energy Equality (2.3). Consider $\phi_\varepsilon(v) = (1/\varepsilon)\log(1+\varepsilon\,|v|^2)$. We have for some constant $C_\varepsilon > 0$, $\phi_\varepsilon(v) \leqslant C_\varepsilon(1+|v|^2)^{1/2}$. Since the solution $0 \leqslant f \in L^\infty([0,\infty); L^1_2(\mathbf{R}^3)) \cap C^1([0,\infty); L^1(\mathbf{R}^3))$, and, by assumption, the kernel B satisfies $\int_{\mathbf{S}^2} B(v-v_*,\omega)\,d\omega \leqslant A_0(1+|v-v_*|^\beta)$, it follows that $Q^\pm(f,f)(v,t)\,\phi_\varepsilon(v) \in L^1(\mathbf{R}^3 \times [0,T])$, $\forall T > 0$. Thus, using the integral form (1.8) of the Boltzmann equation (1.1), we have

$$\int_{\mathbf{R}^3} f(v,t)\,\phi_\varepsilon(v)\,dv = \int_{\mathbf{R}^3} f_0(v)\,\phi_\varepsilon(v)\,dv + \int_0^t d\tau \int_{\mathbf{R}^3} Q(f,f)(v,\tau)\,\phi_\varepsilon(v)\,dv \tag{2.8}$$

and

$$\int_{\mathbf{R}^3} Q(f,f)(v,\tau)\,\phi_\varepsilon(v)\,dv = \frac{1}{2}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\,f(v,\tau)\,f(v_*,\tau) \times [\phi'_\varepsilon + \phi'_{\varepsilon *} - \phi_\varepsilon - \phi_{\varepsilon *}]\,d\omega\,dv_*\,dv$$

Next, using $|v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2$, we have

$$\log\left(\frac{(1+\varepsilon\,|v'|^2)(1+\varepsilon\,|v'_*|^2)}{1+\varepsilon(|v|^2+|v_*|^2)}\right) = \log\left(1 + \frac{\varepsilon^2\,|v'|^2\,|v'_*|^2}{1+\varepsilon(|v|^2+|v_*|^2)}\right)$$

and so

$$\phi'_\varepsilon + \phi'_{\varepsilon *} - \phi_\varepsilon - \phi_{\varepsilon *} = \frac{1}{\varepsilon}\log\left(1 + \frac{\varepsilon^2\,|v'|^2\,|v'_*|^2}{1+\varepsilon(|v|^2+|v_*|^2)}\right) - \frac{1}{\varepsilon}\log\left(1 + \frac{\varepsilon^2\,|v|^2\,|v_*|^2}{1+\varepsilon(|v|^2+|v_*|^2)}\right)$$

If we define

$$J_\varepsilon(v,v_*) = \frac{1}{2\varepsilon}\log\left(1 + \frac{\varepsilon^2\,|v|^2\,|v_*|^2}{1+\varepsilon(|v|^2+|v_*|^2)}\right)\int_{\mathbf{S}^2} B(v-v_*,\omega)\,d\omega$$

then (2.8) is equivalent to

$$\int_{\mathbf{R}^3} f(v,t)\,\phi_\varepsilon(v)\,dv+\int_0^t d\tau \iint_{\mathbf{R}^3\times\mathbf{R}^3} J_\varepsilon(v,v_*)\,f(v,\tau)\,f(v_*,\tau)\,dv_*\,dv$$
$$=\int_{\mathbf{R}^3} f_0(v)\,\phi_\varepsilon(v)\,dv+\int_0^t d\tau \iint_{\mathbf{R}^3\times\mathbf{R}^3} K_\varepsilon(v,v_*)\,f(v,\tau)\,f(v_*,\tau)\,dv_*\,dv \tag{2.9}$$

For functions $J_\varepsilon(v,v_*)$, applying elementary inequality $\log(1+y)\leqslant\sqrt{y}$ $(y\geqslant 0)$ we have

$$\begin{aligned}0&\leqslant J_\varepsilon(v,v_*)\\ &\leqslant\frac{1}{2\varepsilon}\cdot\frac{\varepsilon\,|v|\,|v_*|}{\sqrt{1+\varepsilon(|v|^2+|v_*|^2)}}\cdot A_0(1+|v-v_*|^\beta)\\ &\leqslant\frac{1}{2}\,|v|\,|v_*|\cdot A_0\cdot 2(1+|v|^2)^{\beta/2}\,(1+|v_*|^2)^{\beta/2}\\ &\leqslant A_0(1+|v|^2)(1+|v_*|^2)\end{aligned}$$

Since

$$(1+|v|^2)(1+|v_*|^2)\,f(v,\tau)\,f(v_*,\tau)\in L^1(\mathbf{R}^3\times\mathbf{R}^3\times[0,T])),\qquad \forall T>0$$

and $J_\varepsilon(v,v_*)\to 0(\varepsilon\to 0^+)$ for all $(v,v_*)\in\mathbf{R}^3\times\mathbf{R}^3$, it follows from Lebesgue dominated convergence theorem that

$$\lim_{\varepsilon\to 0^+}\int_0^t d\tau\iint_{\mathbf{R}^3\times\mathbf{R}^3} J_\varepsilon(v,v_*)\,f(v,\tau)\,f(v_*,\tau)\,dv_*\,dv=0,\qquad \forall t\geqslant 0 \tag{2.10}$$

On the other hand, since $\phi_\varepsilon(v)\leqslant|v|^2$ and $\phi_\varepsilon(v)\to|v|^2\,(\varepsilon\to 0^+)$, it follows that

$$\lim_{\varepsilon\to 0^+}\int_{\mathbf{R}^3} f(v,t)\,\phi_\varepsilon(v)\,dv=\int_{\mathbf{R}^3} f(v,t)\,|v|^2\,dv,\qquad \forall t\geqslant 0 \tag{2.11}$$

Therefore the energy equality (2.3) follows from (2.9)–(2.11). ∎

Remarks. 3. From the proof above one sees that for the energy equality (2.3), we do not use the condition $f_0\log f_0\in L^1(\mathbf{R}^3)$, and the kernel $B(z, \omega)$ is only needed to satisfy the following weak angular cutoff condition:

$$\int_{\mathbf{S}^2} B(z, \omega)\, d\omega \leqslant C(1+|z|), \qquad z \in \mathbf{R}^3, \quad C = \text{constant}$$

4. The limit in (2.3) may be interpreted as an energy production because we cannot prove that this limit is always zero. That is, we cannot generally prove that the family of the nonnegative integrands $K_\varepsilon(v, v_*)\, f(v, \tau)\, f(v_*, \tau)$ can be dominated by a function which belongs to $L^1(\mathbf{R}^3 \times \mathbf{R}^3 \times [0, T])$ ($\forall T > 0$), so that we could not use the Lebesgue dominated convergence theorem. In fact, the properties (2.1)–(2.3) in Theorem 1 possess weak stability: Given a sequence $\{f^n\}_{n=1}^{\infty}$ of nonnegative solutions of Eq. (1.1) in the class $L^\infty([0, \infty);\ L^1_2(\mathbf{R}^3)) \cap C^1([0, \infty);\ L^1(\mathbf{R}^3))$ satisfying $\sup_{n \geqslant 1} \sup_{t \geqslant 0} \int_{\mathbf{R}^3} f^n(v, t)(1 + |v|^2 + |\log f^n(v, t)|)\, dv < \infty$, here the collision kernel is the same as in Theorem 1. Then following the weak compactness argument (see ref. 1 or ref. 7), there exist a subsequence $\{f^{n_k}\}_{k=1}^{\infty}$ of $\{f^n\}_{n=1}^{\infty}$ and a nonnegative function $f \in L^\infty([0, \infty);\ L^1_2(\mathbf{R}^3)) \cap C^1([0, \infty);\ L^1(\mathbf{R}^3))$ such that for any $t \geqslant 0$, $f^{n_k}(\cdot, t)$ converge weakly in $L^1(\mathbf{R}^3)$ to $f(\cdot, t)$, and f is a solution of (1.1) with initial datum $f_0 = f(\cdot, 0)$ satisfying (1.6). Thus, by Theorem 1, f holds the entropy identity (2.1) and the energy equality (2.3) (therefore the non-decrease of energy (2.2)). But even if all f^n conserve their energy, we only get

$$\forall t \geqslant 0, \qquad \int_{\mathbf{R}^3} f(v, t)\, |v|^2\, dv \leqslant \liminf_{k \to \infty} \int_{\mathbf{R}^3} f^{n_k}(v, 0)\, |v|^2\, dv$$

Of course, if for some $\tilde{f}_0 \in L^1_2(\mathbf{R}^3)$, $\lim_{n \to \infty} \|f^n(\cdot, 0) - \tilde{f}_0\|_{L^1_2} = 0$, we must have $f(v, 0) = \tilde{f}_0(v)$ a.e. on $\mathbf{R}^3$. In this case, f conserves the energy. Wennberg recently proved that (thanks to his estimate (1.11)) for any given initial datum f_0 satisfying (1.6), there exist many nonnegative solutions f of (1.1) in the class $L^\infty([0, \infty);\ L^1_2(\mathbf{R}^3)) \cap C^1([0, \infty);\ L^1(\mathbf{R}^3))$ which have the same initial datum $f(\cdot, 0) = f_0$ such that the corresponding energy $t \mapsto \int_{\mathbf{R}^3} f(v, t)\, |v|^2\, dv$ are increasing step functions on $[0, \infty)$, especially they satisfy

$$\int_{\mathbf{R}^3} f(v, t)\, |v|^2\, dv > \int_{\mathbf{R}^3} f(v, 0)\, |v|^2\, dv, \qquad t > 0$$

This result has been reported at a conference in Vienna, October 1998. From this result of Wennberg, we see that the restriction in Theorems 2–3 that the solutions conserve the energy (i.e., they are conservative solutions) is also a necessary condition for the local stability (including the uniqueness) and for the equivalence of (2.6) and (2.7). (Note that the condition (2.7) already implies the conservation of energy.) And from the above non-uniqueness result we also understand why the conservative solutions obtained before were all obtained by using suitable sequences of conservative approximate solutions whose initial data converge to the given initial datum with the norm of the smaller class $L^1_2(\mathbf{R}^3)$ rather than with the norm of the class $L^1(\mathbf{R}^3)$ only.

3. ENTROPY IDENTITY

The proof of entropy identity (2.1) is relatively complicated. We first prove some lemmas. In the following, we denote $a \wedge b = \min\{a, b\}$, $(y)^+ = \max\{y, 0\}$.

Lemma 1. Let $f, f_*, f', f'_* \in [0, \infty)$; $\phi, \phi_*, \phi', \phi'_* \in (0, 1]$, and $n \geqslant 1$. Define

$$\begin{aligned}\Gamma^+_n(f', f'_*, \phi', \phi'_*; f, f_*, \phi, \phi_*)\\ =\left[(f'f'_* - ff_*)\log\left(\frac{(f'\wedge n+\phi')(f'_*\wedge n+\phi'_*)}{(f\wedge n+\phi)(f_*\wedge n+\phi_*)}\right)\right]^+\end{aligned} \tag{3.1}$$

$$\begin{aligned}\Gamma^-_n(f', f'_*, \phi', \phi'_*; f, f_*, \phi, \phi_*)\\ =\left[-(f'f'_* - ff_*)\log\left(\frac{(f'\wedge n+\phi')(f'_*\wedge n+\phi'_*)}{(f\wedge n+\phi)(f_*\wedge n+\phi_*)}\right)\right]^+\end{aligned} \tag{3.2}$$

Then

$$\begin{aligned}&\Gamma^+_n(f', f'_*, \phi', \phi'_*; f, f_*, \phi, \phi_*)\\ &\quad\leqslant \Gamma(f'f'_*, ff_*) + 4(f'+\phi')(f'_*+\phi'_*) + 4(f+\phi)(f_*+\phi_*)\\ &\qquad + 2(f'+f'_*)[(f+\phi)\wedge(f_*+\phi_*)] + 2(f+f_*)[(f'+\phi')\wedge(f'_*+\phi'_*)]\end{aligned} \tag{3.3}$$

$$\begin{aligned}&\Gamma^-_n(f', f'_*, \phi', \phi'_*; f, f_*, \phi, \phi_*)\\ &\quad\leqslant 4(f'+\phi')(f'_*+\phi'_*) + 4(f+\phi)(f_*+\phi_*)\\ &\qquad + 2(f'+f'_*)[(f+\phi)\wedge(f_*+\phi_*)] + 2(f+f_*)[(f'+\phi')\wedge(f'_*+\phi'_*)]\end{aligned} \tag{3.4}$$

where $\Gamma(\cdot, \cdot)$ is nonnegative, defined by

$$\Gamma(a, b) = \begin{cases} (a-b)\log\left(\frac{a}{b}\right), & a>0, \quad b>0; \\ +\infty, & a>0, \quad b=0 \text{ or } a=0, \quad b>0; \\ 0, & a=b=0 \end{cases} \tag{3.5}$$

Proof. By symmetry, we may suppose that

$$(f' \wedge n+\phi')(f'_* \wedge n+\phi'_*) \geqslant (f \wedge n+\phi)(f_* \wedge n+\phi_*)$$

We first prove that

$$\begin{aligned} R &:= ff_* \frac{(f' \wedge n+\phi')(f'_* \wedge n+\phi'_*)}{(f \wedge n+\phi)(f_* \wedge n+\phi_*)} \\ &\leqslant (f'+\phi')(f'_*+\phi'_*)+4ff_*+2(f+f_*)[(f'+\phi') \wedge (f'_*+\phi'_*)] \end{aligned} \tag{3.6}$$

The cases of $f, f_* \leqslant n$ and $f, f_* \geqslant n$ are easy since $\phi', \phi'_* \leqslant 1 \leqslant n$. Suppose $f \geqslant n \geqslant f_*$ or $f \leqslant n \leqslant f_*$. Then

$$\begin{aligned} R &\leqslant (f+f_*) \frac{(f' \wedge n+\phi')(f'_* \wedge n+\phi'_*)}{n} \\ &\leqslant 2(f+f_*)[(f'+\phi') \wedge (f'_*+\phi'_*)] \end{aligned}$$

Now we prove (3.3) and (3.4). If $f'f'_* \leqslant ff_*$, then $\Gamma_n^+ = 0$ and, using $\log y < y$ $(y>0)$,

$$\Gamma_n^- \leqslant (ff_* - f'f'_*) \frac{(f' \wedge n+\phi')(f'_* \wedge n+\phi'_*)}{(f \wedge n+\phi)(f_* \wedge n+\phi_*)} \leqslant R$$

If $f'f'_* > ff_* = 0$, then $\Gamma_n^- = 0$ and $\Gamma_n^+ < \infty = \Gamma(f'f'_*, ff_*)$. Finally, if $f'f'_* > ff_* > 0$, then $\Gamma_n^- = 0$ and

$$\begin{aligned} \Gamma_n^+ &= \Gamma(f'f'_*, ff_*) + (f'f'_* - ff_*) \log\left(\frac{ff_*(f' \wedge n+\phi')(f'_* \wedge n+\phi'_*)}{f'f'_*(f \wedge n+\phi)(f_* \wedge n+\phi_*)}\right) \\ &\leqslant \Gamma(f'f'_*, ff_*) + R \end{aligned}$$

Therefore (3.3), (3.4) follows from (3.6). ∎

Lemma 2. Suppose the collision kernel $B(z, \omega)$ satisfy (1.4) and (1.5). Let f, g, and h be nonnegative measurable functions belonging to $L^1_\beta(\mathbf{R}^3)$. Then

$$\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*, \omega)\ f(v)\cdot[g(v')\wedge h(v'_*)]\ d\omega\ dv_*\ dv$$

$$\leqslant 8A_0\ \|f\|_{L^1_\beta}(\|g\|_{L^1_\beta}+\|h\|_{L^1_\beta}) \tag{3.7}$$

where A_0 is the constant (1.5).

Proof. We need the following properties of a general collision gain term (ref. 13): Write $B(z, \omega)=\bar{B}(|z|, |z|^{-1}\ |\langle z, \omega\rangle|)$ and let $F\in C(\mathbf{R}^3\times\mathbf{R}^3)$ be nonnegative. Then

$$\iint_{\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*, \omega)\ F(v', v'_*)\ d\omega\ dv_*$$

$$=2\int_0^{\pi/2}\sin(\theta)\int_{\mathbf{R}^3}\bar{B}(|z|, \cos(\theta))$$

$$\times\left[\int_{\mathbf{S}^1(z)} F(v-\cos(\theta)\ z, v-\sin(\theta)\ |z|\ \omega)\ d^\perp\omega\right]\ dz\ d\theta \tag{3.8}$$

and for all $\theta\in(0, \pi/2)$, $v\in\mathbf{R}^3$,

$$\int_{\mathbf{R}^3}\bar{B}(|z|, \cos(\theta))\left[\int_{\mathbf{S}^1(z)} F(v-\cos(\theta)\ z, v-\sin(\theta)\ |z|\ \omega)\ d^\perp\omega\right]\ dz$$

$$=\int_{\mathbf{R}^3}\bar{B}(|z|, \cos(\theta))\left[\int_{\mathbf{S}^1(z)} F(v-\cos(\theta)\ |z|\ \omega, v-\sin(\theta)\ z)\ d^\perp\omega\right]\ dz \tag{3.9}$$

where for any $z\in\mathbf{R}^3\backslash\{0\}$, $\mathbf{S}^1(z):=\{\omega\in\mathbf{S}^2\mid\omega\perp z\}$, and $d^\perp\omega$ denotes the Lebesgue measure on the circle $\mathbf{S}^1(z)$, i.e.,

$$\int_{\mathbf{S}^1(z)}\varphi(\omega)\ d^\perp\omega$$

$$:=\int_0^{2\pi}\varphi(\cos(\theta)\ \mathrm{i}+\sin(\theta)\ \mathrm{j})\ d\theta,\quad \varphi\in C(\mathbf{S}^2);\qquad \mathrm{i}, \mathrm{j}\in\mathbf{S}^1(z),\quad \mathrm{i}\perp\mathrm{j}$$

Now we prove (3.7). By standard L^1-approximation and Fatou's Lemma with respect to the product measure $d\omega \otimes dv_* \otimes dv$, we may suppose that g, h are also continuous on $\mathbf{R}^3$. Let $F(v, v_*) = g(v) \wedge h(v_*)$. If $\theta \in (0, \pi/4]$, then, by (1.4), the left-hand side of (3.9) is bounded by

$$b(\theta)\, 2\pi \int_{\mathbf{R}^3} (1+|z|^\beta)\, g(v-\cos(\theta)\, z)\, dz$$

$$\leqslant b(\theta)\, 8\pi \int_{\mathbf{R}^3} (1+|z|^\beta)\, g(v-z)\, dz$$

$$\leqslant b(\theta)\, 16\pi(1+|v|^2)^{\beta/2} \int_{\mathbf{R}^3} (1+|v_*|^2)^{\beta/2}\, g(v_*)\, dv_*$$

Similarly, if $\theta \in [\pi/4, \pi/2)$, then the right-hand side of (3.9) is bounded by

$$b(\theta)\, 2\pi \int_{\mathbf{R}^3} (1+|z|^\beta)\, h(v-\sin(\theta)\, z)\, dz$$

$$\leqslant b(\theta)\, 16\pi(1+|v|^2)^{\beta/2} \int_{\mathbf{R}^3} (1+|v_*|^2)^{\beta/2}\, h(v_*)\, dv_*$$

Therefore, by (3.8) and (1.5),

$$\iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v-v_*, \omega)[\, g(v') \wedge h(v'_*)]\, d\omega\, dv_*$$

$$\leqslant 8A_0(1+|v|^2)^{\beta/2}\, (\|g\|_{L^1_\beta} + \|h\|_{L^1_\beta})$$

This yields the estimate (3.7). ∎

Lemma 3. If $f(t)$ is absolutely continuous on $[a, b]$, $G(y)$ is Lipschitz continuous on $[c, d]$ such that $f([a, b]) \subset [c, d]$. Then

$$G(f(t)) = G(f(a)) + \int_a^t G_1(f(\tau))\, \frac{d}{d\tau} f(\tau)\, d\tau, \qquad t \in [a, b]$$

where $G_1(y) = (d/dy)\, G(y)$ a.e. on $[c, d]$.

Proof. See ref. 16, p. 223 Theorem 4.3, p. 263 Theorem 4.9, and note that by assumption of the lemma the function $t \mapsto G(f(t))$ is also absolutely continuous on $[a, b]$. ∎

Proof of Entropy Identity (2.1). The proof of (2.1) requires several steps. First of all we recall that the nonnegative solution f in Theorem 1 is arbitrarily given in $L^\infty([0,\infty); L^1_2(\mathbf{R}^3)) \cap C^1([0,\infty); L^1_0(\mathbf{R}^3))$, and the collision kernel $B(z, \omega)$ satisfies (1.4) and (1.5). This implies first that the set

$$\mathscr{Z} = \{(v, v_*, \omega, t) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2 \times [0,\infty) \mid f(v,t) + f(v_*, t) + f(v', t) + f(v'_*, t) = \infty\}$$

has measure zero with respect to the product measure $dv \otimes dv_* \otimes d\omega \otimes dt$ (or respect to the completion of the measure), and $Q^\pm(f,f)(v,t) \times (1+|v|^2)^{1/2} \in L^1(\mathbf{R}^3 \times [0,T])$ $(\forall T>0)$.

Step 1. Let $\Phi(v) = (1+|v|^2)^{-4}$, $\phi_n(v) = (1/n)\,\Phi(v)$, $n \geqslant 1$. For any fixed v, n, the function $y \mapsto (y + \phi_n(v)) \log(y \wedge n + \phi_n(v))$ is Lipschitz continuous on $[0, R]$ $(\forall R > 0)$ and

$$\frac{d}{dy}\{(y+\phi_n(v))\log(y \wedge n + \phi_n(v))\} = 1 + \log(y \wedge n + \phi_n(v)) - \chi_{\{y>n\}}, \qquad \forall y \in [0,\infty), \quad y \neq n$$

where $\chi_{\{\,\}}$ is the characteristic function, for instance $\chi_{\{a>b\}} = 1$ if $a > b$; $\chi_{\{a>b\}} = 0$ if $a \leqslant b$. On the other hand, from the integral form (1.8) of the Boltzmann equation (1.1), we know that for almost all $v \in \mathbf{R}^3$, $t \mapsto f(v,t)$ is absolutely continuous on $[0,T]$ $(\forall T>0)$. Thus, using Lemma 3 we have for almost all $v \in \mathbf{R}^3$

$$\begin{aligned}(f(v,t)+\phi_n(v))\log(f(v,t)\wedge n+\phi_n(v)) &= (f_0(v)+\phi_n(v))\log(f_0(v)\wedge n+\phi_n(v)) \\ &\quad + \int_0^t \{1+\log(f(v,\tau)\wedge n+\phi_n(v)) - \chi_{\{f(v,\tau)>n\}}\} \\ &\quad \times Q(f,f)(v,\tau)\,d\tau, \qquad t\in[0,\infty)\end{aligned} \tag{3.10}$$

Furthermore we have

$$|\log(f(v,t)\wedge n+\phi_n(v))| \leqslant \log(n+1) + 4\log(1+|v|^2)$$

This insures the existence of the following integrals

$$H_n(f)(t) := \int_{\mathbf{R}^3} (f(v,t)+\phi_n(v)) \log(f(v,t) \wedge n + \phi_n(v))\, dv$$

$$R_n(f)(t) := \int_0^t d\tau \int_{\mathbf{R}^3} Q(f,f)(v,\tau)\, \chi_{\{f(v,\tau)>n\}}\, dv$$

$$\Delta_n^{\pm}(f)(t) := \frac{1}{4}\int_0^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)$$
$$\times \Gamma_n^{\pm}(f', f'_*, \phi'_n, \phi'_{n*}; f, f_*, \phi_n, \phi_{n*})\, d\omega\, dv_*\, dv$$

where $\Gamma_n^{\pm}$ are defined in (3.1), (3.2) and $f_*=f(v_*,\tau)$, $f'=f(v',\tau)$, $f'_*=f(v'_*,\tau)$, etc. Therefore, taking integration for both sides of (3.10) over $v\in\mathbf{R}^3$ and using the identity $y=(y)^+-(-y)^+$ leads to

$$H_n(f)(t) = H_n(f_0) - \Delta_n^+(f)(t) + \Delta_n^-(f)(t) - R_n(f)(t) \tag{3.11}$$

or

$$\Delta_n^+(f)(t) = H_n(f_0) - H_n(f)(t) + \Delta_n^-(f)(t) - R_n(f)(t) \tag{3.12}$$

Step 2. In order that the limit (with respect to n) can be taken into the integrands, we need several estimates. First, it is easily shown that

$$(f(v,t)+\phi_n(v))\, |\log(f(v,t)\wedge n+\phi_n(v))|$$
$$\leqslant f(v,t)(1+|\log f(v,t)|) + 4\Phi(v)(1+|v|^2) \tag{3.13}$$

Next, let

$$\begin{aligned} F(v,v_*,\omega,\tau) = {} & 4(f(v,\tau)+\Phi(v))(f(v_*,\tau)+\Phi(v_*)) \\ & + 4(f(v',\tau)+\Phi(v'))(f(v'_*,\tau)+\Phi(v'_*)) \\ & + 2(f(v,\tau)+f(v_*,\tau))[(f(v',\tau)+\Phi(v')) \wedge (f(v'_*,\tau)+\Phi(v'_*))] \\ & + 2(f(v',\tau)+f(v'_*,\tau))[(f(v,\tau)+\Phi(v)) \wedge (f(v_*,\tau)+\Phi(v_*))] \end{aligned} \tag{3.14}$$

Then, by Lemma 1, we have

$$\Gamma_n^+(f', f'_*, \phi'_n, \phi'_{n*}; f, f_*, \phi_n, \phi_{n*}) \leqslant \Gamma(f'f'_*, ff_*) + F(v,v_*,\omega,\tau) \tag{3.15}$$

$$\Gamma_n^-(f', f'_*, \phi'_n, \phi'_{n*}; f, f_*, \phi_n, \phi_{n*}) \leqslant F(v,v_*,\omega,\tau) \tag{3.16}$$

Now, applying Lemma 2 to functions in (3.14), we obtain

$$\int_0^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\, F(v,v_*,\omega,\tau)\, d\omega\, dv_*\, dv$$

$$= 8\int_0^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)(f(v,\tau)+\Phi(v))$$

$$\times (f(v_*,\tau)+\Phi(v_*))\, d\omega\, dv_*\, dv$$

$$+ 8\int_0^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\ f(v,\tau)$$

$$\times [(f(v',\tau)+\Phi(v'))\wedge (f(v'_*,\tau)+\Phi(v'_*))]\, d\omega\, dv_*\, dv$$

$$\leqslant 16A_0 \int_0^t (\|f(\cdot,\tau)+\Phi\|_{L^1_\beta})^2\, d\tau$$

$$+ 128A_0 \int_0^t \|f(\cdot,\tau)\|_{L^1_\beta}\, \|f(\cdot,\tau)+\Phi\|_{L^1_\beta}\, d\tau$$

$$\leqslant 144A_0 (\sup_{\tau\geqslant 0} \|f(\cdot,\tau)+\Phi\|_{L^1_\beta})^2\, t$$

$$:= C_1(t) < \infty, \qquad t\in[0,\infty) \tag{3.17}$$

Further, let

$$C_2(t) = \int_0^t d\tau \int_{\mathbf{R}^3} |Q(f,f)(v,\tau)|\, dv$$

Then $|R_n(f)(t)| \leqslant C_2(t) < \infty$, and so by (3.11), (3.16) and (3.17) we have for all $t \geqslant 0$,

$$\int_{\mathbf{R}^3} (f(v,t)+\phi_n(v))\, |\log(f(v,t)\wedge n + \phi_n(v))|\, dv$$

$$= H_n(f)(t) + 2\int_{\mathbf{R}^3} (f(v,t)+\phi_n(v))$$

$$\times \log[(f(v,t)+\phi_n(v))^{-1}]\, \chi_{\{f(v,t)+\phi_n(v)<1\}}\, dv$$

$$\leqslant H_n(f_0) + \Delta_n^-(f)(t) + |R_n(f)(t)|$$

$$+2\int_{\mathbf{R}^3} (f(v,t)+\phi_n(v)) \log[(f(v,t)+\phi_n(v))^{-1}]\, \chi_{\{f(v,t)+\phi_n(v)<e^{-|v|^2}\}}\, dv$$

$$+2\int_{\mathbf{R}^3} (f(v,t)+\phi_n(v)) \log[(f(v,t)+\phi_n(v))^{-1}]\, \chi_{\{e^{-|v|^2}\leqslant f(v,t)+\phi_n(v)<1\}}\, dv$$

$$\leqslant \int_{\mathbf{R}^3} \{f_0(v)(1+|\log f_0(v)|) + 4\Phi(v)(1+|v|^2)\}\, dv + \frac{1}{4} C_1(t) + C_2(t)$$

$$+2\int_{\mathbf{R}^3} e^{-(1/2)\,|v|^2}\, dv + 2\int_{\mathbf{R}^3} (f(v,t)+\Phi(v))\, |v|^2\, dv$$

$$:= C_3(t) < \infty \tag{3.18}$$

Step 3. From (3.18) and Fatou's Lemma we see that $f(\cdot, t)\times \log f(\cdot, t) \in L^1(\mathbf{R}^3)$, $\forall t \in [0, \infty)$. This, together with (3.13) and dominated convergence theorem, implies that

$$\lim_{n\to\infty} H_n(f)(t) = H(f)(t), \qquad t \in [0, \infty) \tag{3.19}$$

Also, since $\chi_{\{f(v,\tau)>n\}} \to 0$ $(n\to\infty)$ for a.e. $(v,\tau)\in \mathbf{R}^3\times(0,\infty)$ and $Q(f,f) \in L^1(\mathbf{R}^3\times[0,T])(\forall T>0)$, it follows that

$$\lim_{n\to\infty} R_n(f)(t) = 0, \qquad t\in[0,\infty) \tag{3.20}$$

Next, by definition of $\Gamma_n^{\pm}$ and Γ, it is easily verified that

$$\lim_{n\to\infty} \Gamma_n^+(f', f'_*, \phi'_n, \phi'_{n*}; f, f_*, \phi_n, \phi_{n*})$$
$$= \Gamma(f'f'_*, ff_*) \qquad \text{on} \quad \mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2\backslash\mathscr{Z} \tag{3.21}$$

$$\lim_{n\to\infty} \Gamma_n^-(f', f'_*, \phi'_n, \phi'_{n*}; f, f_*, \phi_n, \phi_{n*})$$
$$= 0 \qquad \text{on} \quad \mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2\backslash\mathscr{Z} \tag{3.22}$$

From (3.16), (3.17) and (3.22) we have

$$\lim_{n\to\infty} \Delta_n^-(f)(t) = 0, \qquad t\in[0,\infty) \tag{3.23}$$

Therefore, by (3.12), (3.19), (3.20) and (3.23),

$$\lim_{n\to\infty} \Delta_n^+(f)(t) = H(f_0) - H(f)(t), \qquad t \in [0, \infty) \tag{3.24}$$

If we define

$$\Delta(f)(t) = \frac{1}{4}\int_0^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*, \omega)\, \Gamma(f'f'_*, ff_*)\, d\omega\, dv_*\, dv$$

then by (3.21), (3,24) and Fatou's Lemma, we obtain

$$\Delta(f)(t) \leqslant H(f_0) - H(f)(t) < \infty, \qquad \forall t \in [0, \infty)$$

This integrability together with the estimates (3.15), (3.17), equalities (3.21), (3.24) and dominated convergence theorem imply that the equality $\Delta(f)(t) = H(f_0) - H(f)(t)$ holds for all $t \in [0, \infty)$. Equivalently, we obtain the entropy identity (2.1):

$$H(f)(t) = H(f_0) - \Delta(f)(t), \qquad t \in [0, \infty) \tag{3.25}$$

Finally, from (3.25) one easily shows that $\sup_{t\geqslant 0} \int_{\mathbf{R}^3} f(v, t)\, |\log f(v, t)|\, dv < \infty$. This completes the proof. ∎

The proof above shows that the classical formal entropy identity (2.1) does actually hold under the conditions in Theorem 1 and therefore it holds the equality of the *entropy production* (ref. 4): for almost all $t \in (0, \infty)$,

$$-\int_{\mathbf{R}^3} Q(f, f) \log f\, dv$$

$$= \frac{1}{4}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*, \omega)(f'f'_* - ff_*) \log\left(\frac{f'f'_*}{ff_*}\right) d\omega\, dv_*\, dv$$

where the left-hand side for general case (as in Theorem 1) can be at least defined by the right-hand side since the right-hand side is finite for a.e. $t \in (0, \infty)$. This may be helpful for further investigation on the entropy production estimates (refs. 4 and 5). Also, it is known that if the collision kernel is given by (1.3), (1.5) with $0 \leqslant \beta \leqslant 1$, then the solution f in Theorem 1 is positive in $\mathbf{R}^3 \times (0, \infty)$. (More precisely, for any $t_0 > 0$, f is bounded pointwise from below by a Maxwellian $c_1 \exp(-c_2 |v|^2)$ on $\mathbf{R}^3 \times [t_0, \infty)$, where $c_1 > 0$, $c_2 > 0$ depend only on the mass, energy, $H(f_0)$ and t_0. See ref. 17.) This insures that in the entropy identity, $(f'f'_* - ff_*) \log[(f'f'_*)/(ff_*)]$ is meaningful. In the general case of (1.4) (with (1.5)), $(f'f'_* - ff_*) \times \log[(f'f'_*)/(ff_*)]$ can be at least understood as $\Gamma(f'f'_*, ff_*)$ (see (3.5)) which was first used in ref. 9 for proving the entropy inequality of inhomogeneous solutions.

4. FURTHER ESTIMATE ON THE MOMENT PRODUCTION

Our improvement for the Wennberg's estimate (1.11) is to replace the exponent s/β by $(s-2)/\beta$. We begin by introducing some notations. Let

$$K(\theta) = \min\{(\cos(\theta))^2, (1-\cos(\theta))^2\}, \qquad \theta \in [0, \pi/2] \tag{4.1}$$

$$A_s = 4\pi \int_0^{\pi/2} b(\theta)[K(\theta)]^{s/2} \sin(\theta)\, d\theta, \qquad s \geqslant 0 \tag{4.2}$$

$$C_\beta(x, y, z) = \left(\frac{1}{2}\right)^{2+\beta} \left(\frac{3}{16\pi}\right)^{\beta/3} \times \min\left\{x, \frac{x^{2+\beta/3}}{y} \exp\left(-\frac{4\beta(y+z+(2\pi)^3/2)}{3x}\right)\right\} \tag{4.3}$$

where $\beta > 0$, $(x, y, z) \in (0, \infty) \times (0, \infty) \times \mathbf{R}$, and the positive function $C_\beta(x, y, z)$ comes from a well known result of Arkeryd (ref. 2, see also ref. 7) i.e., a lower bound:

$$\inf_{(v,t) \in \mathbf{R}^3 \times [0,\infty)} (1+|v|^2)^{-\beta/2} \int_{\mathbf{R}^3} f(v_*, t)\, |v - v_*|^\beta\, dv_* \geqslant C_\beta(\|f_0\|_{L^1_0}, \|f_0\|_{L^1_2}, H(f_0)) \tag{4.4}$$

where $L^1_0 = L^1$ and $f \in L^\infty([0, \infty); L^1_2(\mathbf{R}^3)) \cap C^1([0, \infty); L^1(\mathbf{R}^3))$ is a conservative solution of (1.1) with the initial datum f_0 satisfying (1.6) (therefore f holds the entropy identity (2.1)).

Theorem 4. Assume the kernel $B(z, \omega)$ satisfy (1.3) and (1.5) with $0 < \beta \leqslant 1$. Let f_0 satisfy (1.6). Then there exists a conservative solution f of (1.1) with $f|_{t=0} = f_0$ such that

$$\|f(\cdot, t)\|_{L^1_s} \leqslant \|f_0\|_{L^1_2} \left[\frac{b}{1-\exp(-at)}\right]^{(s-2)/\beta}, \qquad t > 0, \quad s \geqslant 2 \tag{4.5}$$

where

$$a = \beta A_0 \|f_0\|_{L_2^1}, \qquad b = \frac{8^s A_0 \|f_0\|_{L_2^1}}{A_s C_\beta(\|f_0\|_{L_2^1}, \|f_0\|_{L_2^1}, H(f_0))}$$

To prove Theorem 4, we first prove a sharpened version of the Povzner inequality which is also used for proving Theorem 3.

Lemma 4. Let $\varrho(v) = 1 + |v|^2$, $k > 1$, $0 \leqslant \lambda \leqslant 1/2$. Then for all $(v, v_*, \omega) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2$,

$$\begin{aligned}&[\varrho(v')]^k + [\varrho(v'_*)]^k - [\varrho(v)]^k - [\varrho(v_*)]^k \\ &\quad \leqslant 2(2^k - 2)\{[\varrho(v)]^{k-\lambda} [\varrho(v_*)]^\lambda + [\varrho(v)]^\lambda [\varrho(v_*)]^{k-\lambda}\} \\ &\qquad - 4^{-k}(k-1)[K(\theta)]^k [\varrho(v)]^k \end{aligned} \tag{4.6}$$

$$\begin{aligned}&-[\varrho(v) + \varrho(v_*)] \\ &\quad \leqslant \varrho(v') \log \varrho(v') + \varrho(v'_*) \log \varrho(v'_*) - \varrho(v) \log \varrho(v) - \varrho(v_*) \log \varrho(v_*) \\ &\quad \leqslant 2[\varrho(v)\, \varrho(v_*)]^{1/2} - \tfrac{1}{4} K(\theta)\, \varrho(v) \end{aligned} \tag{4.7}$$

where $\theta = \arccos(|v - v_*|^{-1} |\langle v - v_*, \omega \rangle|)$. (For $v = v_*$, we define $\theta = 0$.)

Proof. We first prove (4.6). This relies on the following elementary inequalities:

$$(1+x)^k \leqslant 1 + x^k + (2^k - 2)\, x^{k/2}, \qquad x \in [0, 1], \quad 1 < k \leqslant 2; \tag{4.8}$$

$$(1+x)^k \leqslant 1 + x^k + (2^k - 2)\, x, \qquad x \in [0, 1], \quad k \geqslant 2 \tag{4.9}$$

(4.8) can be proven by showing that the function $x \mapsto (1+x)^k x^{-k/2} - x^{-k/2} - x^{k/2}$ is increasing on (0,1]. (4.9) is easy. Now let $\varrho = \varrho(v)$, $\varrho_* = \varrho(v_*)$, $\varrho' = \varrho(v')$, $\varrho'_* = \varrho(v'_*)$, and let $D_k = (\varrho')^k + (\varrho'_*)^k - (\varrho)^k - (\varrho_*)^k$. By identity $\varrho' + \varrho'_* = \varrho + \varrho_*$, we have

$$D_k = (\varrho + \varrho_*)^k - (\varrho)^k - (\varrho_*)^k - [(\varrho' + \varrho'_*)^k - (\varrho')^k - (\varrho'_*)^k] \tag{4.10}$$

Using (4.8) and (4.9), we obtain

$$(\varrho + \varrho_*)^k - (\varrho)^k - (\varrho_*)^k \leqslant (2^k - 2) \max\{(\varrho)^{k-\lambda} (\varrho_*)^\lambda, (\varrho)^\lambda (\varrho_*)^{k-\lambda}\}$$

On the other hand, it is easily shown that

$$(\varrho' + \varrho'_*)^k - (\varrho')^k - (\varrho'_*)^k \geqslant (k-1) \min\{\varrho', \varrho'_*\}$$

Thus

$$D_k \leqslant (2^k-2)\max\{(\varrho)^{k-\lambda}(\varrho_*)^{\lambda}, (\varrho)^{\lambda}(\varrho_*)^{k-\lambda}\} - (k-1)\min\{(\varrho')^k, (\varrho'_*)^k\} \tag{4.11}$$

This implies first that (4.6) holds for $K(\theta)=0$. Now suppose that $K(\theta)>0$. Let

$$M(\theta)=\max\left\{\frac{1+\cos(\theta)}{\cos(\theta)}, \frac{\cos(\theta)}{1-\cos(\theta)}\right\}$$

If $|v| \geqslant 2M(\theta)\,|v_*|$, then, by (1.2),

$$|v'| \geqslant |v| - |v-v_*|\cos(\theta) \geqslant |v|\,(1-\cos(\theta)) - |v_*|\cos(\theta) \geqslant \tfrac{1}{2}(1-\cos(\theta))\,|v|$$

$$|v'_*| \geqslant |v-v_*|\cos(\theta) - |v_*| \geqslant |v|\cos(\theta) - |v_*|\,(\cos(\theta)+1) \geqslant \tfrac{1}{2}\cos(\theta)\,|v|$$

These imply $\varrho' \geqslant (1/4)\,K(\theta)\,\varrho$ and $\varrho'_* \geqslant (1/4)\,K(\theta)\,\varrho$ so that by (4.11) we have

$$D_k \leqslant (2^k-2)\max\{(\varrho)^{k-\lambda}(\varrho_*)^{\lambda}, (\varrho)^{\lambda}(\varrho_*)^{k-\lambda}\} - 4^{-k}(k-1)[K(\theta)]^k(\varrho)^k \tag{4.12}$$

If $|v| \leqslant 2M(\theta)\,|v_*|$, then $\varrho \leqslant (2M(\theta))^2\,\varrho_*$. Since $0 \leqslant \lambda \leqslant k/2$, $K(\theta)\,M(\theta) \leqslant 3/4$ and $k-1 \leqslant 2^k-2$, this implies that $4^{-k}(k-1)[K(\theta)]^k(\varrho)^k \leqslant 4^{-k}(2^k-2)(3/2)^k(\varrho)^{k-\lambda}(\varrho_*)^{\lambda}$. Therefore, using (4.11) again we obtain

$$\begin{aligned} D_k \leqslant{}& (2^k-2)(1+(3/8)^k)\max\{(\varrho)^{k-\lambda}(\varrho_*)^{\lambda}, (\varrho)^{\lambda}(\varrho_*)^{k-\lambda}\} \\ &- 4^{-k}(k-1)[K(\theta)]^k(\varrho)^k \end{aligned} \tag{4.13}$$

which gives the inequality (4.6). To prove (4.7), we write D_k as (using $\varrho'+\varrho'_*=\varrho+\varrho_*$ and, by continuity, assume that $c>0$)

$$\begin{aligned} D_k ={}& \varrho'[(\varrho')^{k-1}-1] + \varrho'_*[(\varrho'_*)^{k-1}-1] \\ &- \varrho[(\varrho)^{k-1}-1] - \varrho_*[(\varrho_*)^{k-1}-1] \end{aligned}$$

and choose $\lambda=1/2$. Then, dividing both side of (4.13) by $k-1$ and letting $k\to 1^+$ leads to the the second inequality in (4.7). The first inequality in (4.7) is easily derived from the following identity (use $\varrho'+\varrho'_*=\varrho+\varrho_*$):

$$\begin{aligned} &\varrho'\log\varrho' + \varrho'_*\log\varrho'_* - \varrho\log\varrho - \varrho_*\log\varrho_* \\ &= \varrho\log(1+\varrho_*/\varrho) + \varrho_*\log(1+\varrho/\varrho_*) - \varrho'\log(1+\varrho'_*/\varrho') - \varrho'_*\log(1+\varrho'/\varrho'_*) \end{aligned}$$

and from the elementary inequality

$$\varrho' \log(1+\varrho'_*/\varrho') + \varrho'_* \log(1+\varrho'/\varrho'_*) \leqslant \varrho'_* + \varrho' = \varrho_* + \varrho$$

This proves the lemma. ∎

Proof of Theorem 4. We will use an approximation process. For any positive integer n, let $f_0^n(v) = f_0(v)\exp(-|v|^2/n)$. Following the weak convergence argument together with the Gronwall inequality and the Povzner inequality, there exists a conservative solution f^n of (1.1) such that $f^n|_{t=0} = f_0^n$ and, for all $s>2$ and all $T>0$, f^n belongs to $L^\infty([0,T]; L_s^1(\mathbf{R}^3)) \cap C^1([0,T]; L_s^1(\mathbf{R}^3))$ (see ref. 1 or ref. 7). Since f^n conserves the mass and energy, and $\|f_0^n\|_{L_2^1} \leqslant \|f_0\|_{L_2^1}$, we have (by Hölder inequality)

$$[\|f^n(\cdot,t)\|_{L_s^1}]^{1+\beta/(s-2)} \leqslant [\|f_0\|_{L_2^1}]^{\beta/(s-2)} \|f^n(\cdot,t)\|_{L_{s+\beta}^1}, \qquad t \geqslant 0, \quad s>2 \tag{4.14}$$

The following proof follows Wennberg's argument: By Lemma 4 (taking $k=s/2$, $\lambda=\beta/2$), (4.4) and (4.14) we have

$$\frac{d}{dt}\|f^n(\cdot,t)\|_{L_s^1} \leqslant a_s(f_0)\,\|f^n(\cdot,t)\|_{L_s^1} - b_s(f_0^n)[\|f^n(\cdot,t)\|_{L_s^1}]^{1+\beta/(s-2)}, \qquad t\geqslant 0, \quad s>2$$

where

$$a_s(f_0) = 2(2^{s/2}-2)\,A_0\,\|f_0\|_{L_2^1},$$
$$b_s(f_0^n) = [(s/2)-1]\,2^{-s-1}[\|f_0\|_{L_2^1}]^{-\beta/(s-2)}\,A_s C_\beta(\|f_0^n\|_{L_0^1}, \|f_0^n\|_{L_2^1}, H(f_0^n))$$

These imply

$$\|f^n(\cdot,t)\|_{L_s^1} \leqslant \left[\frac{\exp\left(\frac{\beta}{s-2}a_s(f_0)\,t\right)}{[\|f_0^n\|_{L_s^1}]^{-\beta/(s-2)} + \frac{b_s(f_0^n)}{a_s(f_0)}\left[\exp\left(\frac{\beta}{s-2}a_s(f_0)\,t\right)-1\right]}\right]^{(s-2)/\beta}, \quad t\geqslant 0, \quad s>2 \tag{4.15}$$

On the other hand, the Boltzmann H-theorem (i.e., the entropy inequality) implies that $\sup_{n\geqslant 1}\sup_{t\geqslant 0}\int_{\mathbf{R}^3} f^n(v,t)(1+|v|^2+|\log f^n(v,t)|)\,dv < \infty$.

Thus there exists a subsequence $\{f^{n_k}\}_{k=1}^{\infty}$ of $\{f^n\}_{n=1}^{\infty}$ such that for any $t \geqslant 0$, $\{f^{n_k}(\cdot, t)\}_{k=1}^{\infty}$ converges weakly in $L^1(\mathbf{R}^3)$ to a function $f(\cdot, t)$, and f is a conservative solution of (1.1) satisfying $f|_{t=0} = f_0$ since $\lim_{n\to\infty} \|f_0^n - f_0\|_{L_2^1} = 0$. Moreover, since $f_0^n \leqslant f_0$ and $C_\beta(x, y, z)$ is continuous in the open set $(0, \infty) \times (0, \infty) \times \mathbf{R}$, it follows that

$$\lim_{n\to\infty} C_\beta(\|f_0^n\|_{L_0^1}, \|f_0^n\|_{L_2^1}, H(f_0^n)) = C_\beta(\|f_0\|_{L_0^1}, \|f_0\|_{L_2^1}, H(f_0))$$

and so $\lim_{n\to\infty} b_s(f_0^n) = b_s(f_0)$. Therefore, by (4.15) and the weak convergence, we obtain

$$\|f(\cdot, t)\|_{L_s^1} \leqslant \left[\frac{a_s(f_0)}{b_s(f_0)[1 - \exp(-(\beta/(s-2))\, a_s(f_0)\, t)]}\right]^{(s-2)/\beta}, \quad t > 0,\ s > 2$$

This implies the estimate (4.5) because $(\beta/(s-2))\, a_s(f_0) \geqslant \beta A_0 \|f_0\|_{L_2^1}$ and

$$\frac{a_s(f_0)}{b_s(f_0)} \leqslant [\|f_0\|_{L_2^1}]^{\beta/(s-2)} \frac{8^s A_0 \|f_0\|_{L_2^1}}{A_s C_\beta(\|f_0\|_{L_0^1}, \|f_0\|_{L_2^1}, H(f_0))} \quad \blacksquare$$

Remark. From the estimates (4.15) of $\{f^n\}_{n=1}^{\infty}$, one immediately obtains a global result of Elmroth: If the initial datum $f_0 \in L_s^1(\mathbf{R}^3)$ for some $s > 2$, then the conservative solution $f \in L^\infty([0, \infty); L_s^1(\mathbf{R}^3))$.

5. LOCAL STABILITY

In this section we prove Theorem 2. It suffices to prove that the conservative solution f obtained in Theorem 4 holds the inequality (2.4) for all conservative solutions g. In fact, this implies the uniqueness of conservative solutions. We first prove that the non-decreasing function $\Psi_f(\cdot)$ defined by (2.5) is continuous on $[0, \infty)$. This needs a generalized dominated convergence theorem (see ref. 16, Theorem 3.4):

Lemma 5. Let $\{F_n\}_{n=1}^{\infty}$ and $\{G_n\}_{n=1}^{\infty}$ be two sequences in $L^1(\mathbf{R}^3)$ which converge a.e. to the functions F and G, respectively. Suppose $|F_n| \leqslant G_n$ and

$$\lim_{n\to\infty} \int_{\mathbf{R}^3} G_n(v)\, dv = \int_{\mathbf{R}^3} G(v)\, dv < \infty$$

Then

$$\lim_{n\to\infty} \int_{\mathbf{R}^3} F_n(v)\, dv = \int_{\mathbf{R}^3} F(v)\, dv$$

Now we prove that $\Psi_f(r)$ is continuous at $r=0$. Since $\Psi_f(r)$ is monotone, it suffices to show that $\lim_{n\to\infty} \Psi_f(1/n)=0$. By definition of $\Psi_f(r)$, for any $n\geqslant 1$ there exists $t_n\in[0, 1/n]$ such that

$$\Psi_f\left(\frac{1}{n}\right)\leqslant\int_{|v|>\sqrt{n}} f(v, t_n)(1+|v|^2)\, dv+\frac{1}{n} \tag{5.1}$$

From the integral form (1.8) of the Boltzmann equation (1.1) we know that for a.e. $v\in\mathbf{R}^3$, the function $t\mapsto f(v, t)$ is continuous on $[0, \infty)$. Thus $\lim_{n\to\infty} f(v, t_n)(1+|v|^2)\,\chi_{\{|v|>\sqrt{n}\}}=0$ and $\lim_{n\to\infty} f(v, t_n)(1+|v|^2)=f_0(v)(1+|v|^2)$ hold for a.e. $v\in\mathbf{R}^3$. Moreover, since f is a conservative solution, we have $\int_{\mathbf{R}^3} f(v, t_n)(1+|v|^2)\, dv=\int_{\mathbf{R}^3} f_0(v)(1+|v|^2)\, dv$ for all $n\geqslant 1$. Thus, by Lemma 5, the right hand side of (5.1) and therefore $\Psi_f(1/n)$ tend to zero as $n\to\infty$. Similarly we can prove that $\Psi_f(r)$ is also continuous at any $r>0$.

Let

$$U_\kappa(t)=\|g(\cdot, t)-f(\cdot, t)\|_{L^1_\kappa}, \qquad t\geqslant 0, \quad 0\leqslant\kappa\leqslant 2$$

We prove that there exist constants C and c which depend only on f_0, β, and the angular function $b(\cdot)$, such that for any conservative solution g of (1.1),

$$U_2(t)\leqslant C[\,U_2(0)+\sqrt{U_2(0)}+\Psi_f(U_2(0))]\, e^{ct}, \qquad t\geqslant 0 \tag{5.2}$$

For convenience, in the following, the same letter C will denote different such constants. If $U_2(0)\geqslant 1$, then the conservation of the mass and energy implies that

$$U_2(t)\leqslant \|g_0\|_{L^1_2}+\|f_0\|_{L^1_2}\leqslant (1+2\,\|f_0\|_{L^1_2})\, U_2(0), \qquad t\geqslant 0 \tag{5.3}$$

where $g_0=g|_{t=0}$. Therefore, in the following, we assume that $U_2(0)<1$ (which implies $\|g_0\|_{L^1_2}\leqslant 1+\|f_0\|_{L^1_2}$ and $U_2(t)\leqslant 1+2\,\|f_0\|_{L^1_2}$ for all $t\geqslant 0$). We need three inequalities: For any $0<r\leqslant 1$,

$$U_2(t)\leqslant U_2(r)+C\int_r^t (1-e^{-a\tau})^{-1}\, U_1(\tau)\, d\tau, \qquad t\geqslant r \tag{5.4}$$

$$U_2(t)\leqslant U_2(0)+\frac{4}{\sqrt{r}}\, U_1(t)+2\Psi_f(r), \qquad 0\leqslant t\leqslant r \tag{5.5}$$

$$U_1(t)\leqslant U_1(0)+C\int_0^t U_2(\tau)\, d\tau, \qquad t\geqslant 0 \tag{5.6}$$

(5.6) is obvious since $0<\beta\leqslant 1$. (5.5) follows from the identity $|g-f|=g-f+2(f-g)^+$ (where $(y)^+=\max\{y,0\}$), the conservation of mass and energy, and the definition (2.5) of $\Psi_f(\cdot)$. Also, for (5.4), we have

$$U_2(t)=\|g(\cdot,r)\|_{L^1_2}-\|f(\cdot,r)\|_{L^1_2}+2\,\|[f(\cdot,t)-g(\cdot,t)]^+\|_{L^1_2},\qquad t\geqslant r \tag{5.7}$$

Then applying the integral form (1.8) of the equation (1.1), we have for a.e. $v\in\mathbf{R}^3$,

$$\begin{aligned}
&[f(v,t)-g(v,t)]^+\\
&\quad=[f(v,r)-g(v,r)]^++\int_r^t d\tau\iint_{\mathbf{R}^3\times\mathbf{S}^2}B(v-v_*,\omega)\\
&\qquad\times\Big\{[f(v',\tau)\,f(v'_*,\tau)-g(v',\tau)\,g(v'_*,\tau)]\\
&\qquad-[f(v,\tau)\,f(v_*,\tau)-g(v,\tau)\,g(v_*,\tau)]\Big\}\chi_{\{f(v,\tau)>g(v,\tau)\}}\,d\omega\,dv_*
\end{aligned}\tag{5.8}$$

Next, by the nonnegativity of f and g, it is easily shown that (ref. 14)

$$\begin{aligned}
&\{(f'f'_*-g'g'_*)-(ff_*-gg_*)\}\chi_{\{f>g\}}\\
&\qquad\leqslant(f'f'_*-g'g'_*)^+-(ff_*-gg_*)^++f\,|g_*-f_*|
\end{aligned}\tag{5.9}$$

Since $(ff_*-gg_*)^+\leqslant ff_*$, and the solution f satisfies the moment estimate (4.5) (choose $s=2+\beta$) which implies that for $t\geqslant r(>0)$

$$\begin{aligned}
&\int_r^t d\tau\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}B(v-v_*,\omega)\;ff_*(1+|v|^2)\,d\omega\,dv\,dv_*\\
&\qquad=A_0\int_r^t d\tau\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}f(v,\tau)\;f(v_*,\tau)(1+|v|^2)\,|v-v_*|^\beta\,dv\,dv_*\\
&\qquad\leqslant A_0\int_r^t\|f(\cdot,\tau)\|_{L^1_{2+\beta}}\,\|f(\cdot,\tau)\|_{L^1_\beta}\,d\tau\\
&\qquad\leqslant A_0(\|f_0\|_{L^1_2})^2\int_r^t\frac{b}{1-\exp(-a\tau)}\,d\tau<\infty
\end{aligned}$$

it follows from $|v'|^2+|v'_*|^2=|v|^2+|v_*|^2$ that

$$\int_r^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)(f'f'_*-g'g'_*)^+\,(1+|v|^2)\,d\omega\,dv\,dv_*$$

$$=\int_r^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)(ff_*-gg_*)^+\,(1+|v|^2)\,d\omega\,dv\,dv_*<\infty$$

Therefore by (5.8) and (5.9),

$$\|[f(\cdot,t)-g(\cdot,t)]^+\|_{L^1_2}$$
$$\leqslant \|[f(\cdot,r)-g(\cdot,r)]^+\|_{L^1_2}$$
$$+\int_r^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\; f\,|g_*-f_*|(1+|v|^2)\,d\omega\,dv\,dv_*$$
$$\leqslant \|[f(\cdot,r)-g(\cdot,r)]^+\|_{L^1_2}+A_0\int_r^t \|f(\cdot,\tau)\|_{L^1_{2+\beta}}\,\|g(\cdot,\tau)-f(\cdot,\tau)\|_{L^1_\beta}\,d\tau$$
$$\leqslant \|[f(\cdot,r)-g(\cdot,r)]^+\|_{L^1_2}+C\int_r^t (1-e^{-a\tau})^{-1}\,U_1(\tau)\,d\tau,\qquad t\geqslant r$$

This estimate together with (5.7) gives (5.4). In (5.4), choose $r=1$. Then, since $U_1(\cdot)\leqslant U_2(\cdot)$, it follows from Gronwall inequality that

$$U_2(t)\leqslant U_2(1)\,e^{c(t-1)},\qquad t\geqslant 1 \tag{5.10}$$

Now let $r>0$ satisfy $U_2(0)\leqslant r\leqslant 1$, and let $U^*(r)=\sup_{0\leqslant t\leqslant r} U_2(t)$. Then using (5.4), (5.6) and Fubini theorem we have

$$U_2(t)\leqslant U_2(r)+C\int_r^t \frac{1}{\tau}\,U_1(\tau)\,d\tau$$
$$\leqslant U_2(r)+CU_1(0)\,|\log r|+C\int_r^t \frac{1}{\tau}\int_0^\tau U_2(\sigma)\,d\sigma\,d\tau$$
$$\leqslant U^*(r)+Cr\,|\log r|+C\int_0^t U_2(\sigma)\,|\log\sigma|\,d\sigma,\qquad t\in[r,1] \tag{5.11}$$

Since $U_2(t)$ also holds the last inequality in (5.11) for $t \in [0, r]$, it follows from Gronwall inequality that

$$U_2(t) \leqslant [U^*(r) + Cr\,|\log r|]\, e^C, \qquad t \in [0, 1] \tag{5.12}$$

For $U^*(r)$, we have, by (5.5) and (5.6),

$$U^*(r) \leqslant U_2(0) + \frac{4}{\sqrt{r}} \left[U_1(0) + C \int_0^r U_2(\tau)\, d\tau \right] + 2\Psi_f(r) \leqslant C[r + \sqrt{r} + \Psi_f(r)]. \tag{5.13}$$

Therefore, combining (5.13), (5.12), (5.10) with (5.3) we obtain

$$U_2(t) \leqslant C[r + \sqrt{r} + \Psi_f(r)]\, e^{ct}, \qquad t \geqslant 0, \quad r > 0, \quad r \geqslant U_2(0)$$

This gives the estimate (5.2) by taking $r = U_2(0)$ for $U_2(0) > 0$ and by letting $r \to 0^+$ for $U_2(0) = 0$, respectively. The proof is completed. ∎

Since the local stability implies the uniqueness, we obtain the following

Corollary of Theorem 2. Under the condition in Theorem 2 (i.e., the kernel B satisfy (1.3) and (1.5) with $0 < \beta \leqslant 1$), the moment estimate (4.5) holds for all conservative solutions f of (1.1) provided that their initial data f_0 satisfy (1.6).

6. PROOF OF THEOREM 3

Let us define

$$M(f_0) = M(f)(0),$$

$$M(f)(t) = \int_{\mathbf{R}^3} f(v, t)(1 + |v|^2) \log(1 + |v|^2)\, dv, \qquad t \in [0, \infty)$$

Since $(1 + |v|^2) \log(1 + |v|^2) \leqslant 10(1 + |v|^2 \log^+ |v|)$ and $f_0 \in L^1_2(\mathbf{R}^3)$, the integrability (2.6) is equivalent to $M(f_0) < \infty$. We first prove that (2.6) implies (2.7). For every integer $n \geqslant 1$, let $f^n_0(v) = f_0(v) \exp(-(1/n)\,|v|^2)$, and let f^n be the unique conservative solution with the initial datum f^n_0. Then, by Elmroth's result (ref. 10) (or using (4.15)), for any $s_1 > 4$, f^n belong to $L^\infty([0, \infty); L^1_{s_1}(\mathbf{R}^3)) \cap C^1([0, \infty); L^1(\mathbf{R}^3))$ for all $n \geqslant 1$. Applying the integral form (1.8) of the equation (1.1) and Lemma 4 we have with $\phi(v) = (1 + |v|^2) \log(1 + |v|^2)$,

$$M(f^n)(t) = M(f_0^n) + \frac{1}{2}\int_0^t d\tau \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*, \omega)$$

$$\times f^n f_*^n(\phi' + \phi'_* - \phi - \phi_*)\, d\omega\, dv_*\, dv$$

$$\leqslant M(f_0^n) + A_0 \int_0^t (\|f^n(\cdot,\tau)\|_{L^1_{1+\beta}})^2\, d\tau$$

$$-\frac{1}{8}A_2 \int_0^t d\tau \int_{\mathbf{R}^3} f^n(v,\tau)(1+|v|^2)$$

$$\times\left[\int_{\mathbf{R}^3} f^n(v_*,\tau)\,|v-v_*|^\beta\, dv_*\right] dv, \qquad t\in[0,\infty) \tag{6.1}$$

where A_0 and A_2 are the positive constants defined in (4.1) and (4.2). Since f^n are conservative solutions, the lower bound (4.4) gives

$$\int_{\mathbf{R}^3} f^n(v_*,\tau)\,|v-v_*|^\beta\, dv_*$$

$$\geqslant C_\beta(\|f_0^n\|_{L^1_0}, \|f_0^n\|_{L^1_2}, H(f_0^n))(1+|v|^2)^{\beta/2}, \qquad v\in\mathbf{R}^3 \tag{6.2}$$

If we define

$$C_B(f_0) = \tfrac{1}{8}A_2\cdot C_\beta(\|f_0\|_{L^1_0}, \|f_0\|_{L^1_2}, H(f_0))$$

then by (6.1), (6.2) and $f_0^n \leqslant f_0$ we obtain

$$\int_0^t \|f^n(\cdot,\tau)\|_{L^1_{2+\beta}}\, d\tau \leqslant \frac{1}{C_B(f_0^n)}(M(f_0) + A_0(\|f_0\|_{L^1_2})^2\, t), \qquad t\in[0,\infty) \tag{6.3}$$

On the other hand, since $\lim_{n\to\infty}\|f_0^n - f_0\|_{L^1_2} = 0$, it follows from Theorem 2 that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|f^n(\cdot,t) - f(\cdot,t)\|_{L^1_2} = 0 \qquad \text{for all} \quad T>0$$

Also, by the continuity and positivity of the function $C_\beta(x,y,z)$ (see (4.3)) we have

$$\lim_{n\to\infty} C_B(f_0^n) = C_B(f_0) > 0$$

Therefore by (6.3) and Fatou's Lemma, we obtain a weak moment production:

$$\int_0^t \|f(\cdot,\tau)\|_{L^1_{2+\beta}}\,d\tau \leqslant \frac{1}{C_B(f_0)}\,(M(f_0)+A_0(\|f_0\|_{L^1_2})^2\,t)<\infty, \qquad t\in[0,\infty)$$

This proves (2.7). Next, we prove that (2.7) implies (2.6). Suppose that f satisfy (2.7). Since f is a conservative solution, it follows from corollary of Theorem 2 that for any $n\geqslant 1$, f belongs to $L^\infty([1/n,\infty); L^1_4(\mathbf{R}^3))$ i.e., $\exists C_n<\infty$, such that $\|f(\cdot,t)\|_{L^1_4}=\int_{\mathbf{R}^3} f(v,t)(1+|v|^2)^2\,dv\leqslant C_n$, $\forall t\geqslant 1/n$. This implies that $M(f)(1/n)<\infty$ $(n=1,2,...)$ and therefore as above using Lemma 4 (with $\phi=(1+|v|^2)\log(1+|v|^2)$),

$$\begin{aligned} M(f)(1) &= M(f)\left(\frac{1}{n}\right)+\frac{1}{2}\int_{1/n}^1 dt \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega) \\ &\quad\times ff_*(\phi'+\phi'_*-\phi-\phi_*)\,d\omega\,dv_*\,dv \\ &\geqslant M(f)\left(\frac{1}{n}\right)-\frac{1}{2}\int_{1/n}^1 dt \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega) \\ &\quad\times ff_*(2+|v|^2+|v_*|^2)\,d\omega\,dv_*\,dv \end{aligned}$$

Thus

$$\begin{aligned} M(f)\left(\frac{1}{n}\right) &\leqslant M(f)(1)+A_0\int_{1/n}^1 dt \iint_{\mathbf{R}^3\times\mathbf{R}^3} f(v,t) \\ &\quad\times f(v_*,t)(1+|v|^2)\,|v-v_*|^\beta\,dv_*\,dv \\ &\leqslant M(f)(1)+A_0\,\|f_0\|_{L^1_2}\int_0^1 dt\int_{\mathbf{R}^3} f(v,t)(1+|v|^2)^{(2+\beta)/2}\,dv \end{aligned} \tag{6.4}$$

Since $\lim_{n\to\infty} f(v,1/n)=f_0(v)$ a.e. $v\in\mathbf{R}^3$, (6.4) together with Fatou's Lemma imply $M(f_0)\leqslant \liminf_{n\to\infty} M(f)(1/n)<\infty$. This proves (2.6). ∎

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